Finite-dimensional observer-based ISS and L^2-gain control of parabolic PDEs

Rami Katz & **Emilia Fridman**

School of Electrical Engineering, Tel-Aviv University

Outline

1. Introduction: delays, spatial & modal decomposition
2. Constructive finite-dimensional observer-based control
3. Delayed implementation
4. Sampled-data implementation
5. Semilinear PDEs

Plan

Effects of delay on stability of PDEs

For **PDEs** arbitrarily **small delays** may **destabilize** the system

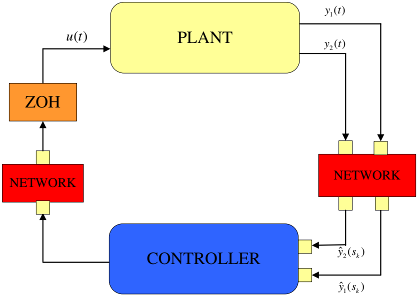
[Datko, SICON'88], [Logemann et al., SICON'96], [Wang, Guo & Krstic, SICON'11]

- The stability of wave eq. is not robust w.r.t. arbitrary small delay:

$$\begin{aligned} &z_{tt}(\xi,t) = z_{\xi\xi}(\xi,t), \quad \xi \in (0,1), \\ &z(0,t) = 0, \quad z_{\xi}(1,t) = -z_t(1,t-h) \end{aligned}$$

- For $h = 0$ all solutions are zero for $t \geq 2$!
- For arbitrary small $h > 0$ the system has unbounded solutions

Networked control systems are systems, where sensors, controller and actuators *exchange data via communication network*.



Benefits: long distance estimation/control, etc.
Imperfections: variable sampling + delays + ...

Motivation: network-based control of PDEs

- Chemical reactors
- Air-polluted areas
- Multi-agents

Figure 1: 800 drone show in Nanchang: multi-agent deployment

Spatial decomposition

Introduced in [Fridman & Blighovsky, Aut '12] for the heat equation

$$z_t(x,t) = z_{xx}(x,t) + \phi\,(z,x,t)\,z(x,t) + \sum_{j=1}^{N} b_j(x)u_j(t), \quad z_x(0,t) = z_x(l,t) = 0$$

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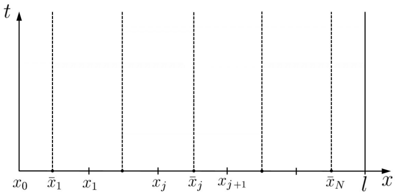

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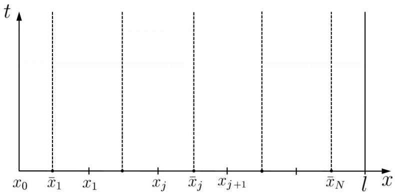

Point measurements:

$$y_j(t) = z(\bar{x}_j, t_k),\ \bar{x}_j = \frac{x_{j-1} + x_j}{2},\ t \in [t_k, t_{k+1})$$

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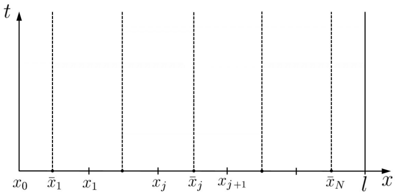

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Static output-feedback: sampled-data via ZOH

$$u_j(t) = -Kz(\bar{x}_j, t_k), \; t \in [t_k, t_{k+1}),$$
$$b_j(x) = \chi_{[x_j, x_{j+1})}(x).$$

Spatial decomposition

Drawback: many actuators covering (almost) all domain & many sensors.

Challenges:

- Few actuators & sensors
- Point (e.g. boundary) control & measurement

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[Karafyllis & Krstic, Aut'18] introduced sampled-data boundary control for heat eq via modal decomposition - state-feedback

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Crucial - explicit estimates on all quantities of interest.

Plan

Finite-dim. observer-based control - problem formulation

In [Katz & Fridman, Aut'20]:

$$\begin{aligned} &z_t(x,t) = \partial_x \left(p(x) z_x(x,t)\right) + \left(q_c - q(x)\right) z(x,t) + b(x)u(t), \;\; t \geq 0, \\ &z_x(0,t) = z(1,t) = 0, \quad y(t) = z(0,t). \end{aligned}$$

- $p \in C^2[0,1], \;\; q \in C^1[0,1]$ satisfying

$$0 < p_* \leq p(x) \leq p^*, \;\; 0 \leq q(x) \leq q^*, \quad x \in [0,1]$$

- $b \in H^1(0,1), \;\; b(1) = 0$
- Non-local actuation and boundary measurement

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For simplicity, consider $p(x) \equiv 1$, $q(x) \equiv 0$ and $q_c = q$.

Finite-dim. observer-based control - modal decomposition

Sturm-Liouville problem:

$$\phi''(x) + \lambda\phi(x) = 0, \quad 0 < x < 1; \quad \phi'(0) = 0, \quad \phi(1) = 0.$$

- $\to$ Corresponding eigenvalues $\lambda_1 < \lambda_2 < ...$ satisfy $\lim_{n\to\infty} \lambda_n = \infty$.
- $\to$ Complete and orthonormal (in $L^2(0,1)$) sequence of eigenfunctions.

Here $\lambda_n = \pi^2 \left(n - \frac{1}{2}\right)^2$, $\phi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}x), \quad n \geq 1$.

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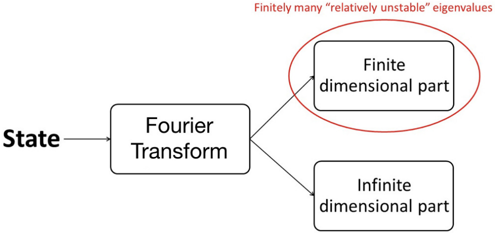
Modal decomposition:

$$z(x,t) = \sum_{n=1}^{\infty} z_n(t)\phi_n(x), \ z_n(t) := \langle z(\cdot,t), \phi_n\rangle, \ t \geq 0.$$

Differentiation of $\langle z(\cdot,t), \phi_n\rangle$ + integration by parts:

$$\begin{aligned} &\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t), \\ &z_n(0) = \langle z_0, \phi_n\rangle =: z_{0,n}, \quad b_n = \langle b, \phi_n\rangle . \end{aligned}$$

Modal decomposition



- Popular in 80s - [Curtain, TAC '82, '92], [Balas, JMAA '88].
- Popular again because of
  - *robustness to sampling/delay*:
    state-feedback [Karafyllis & Krstic, Aut'18],
    finite-dimensional observer [Selivanov & Fridman, TAC'19]
  - *input delay compensation*:
    state-feedback [Prieur & Trelat, TAC'18], [Lhachemi et al, Aut'19]

Works on observer-based control via modal decomposition

- Finite-dimensional observer-based control: bounded control & observation operators
  1. [Curtain, TAC'82] - restrictive assumptions ($b_n = 0, \ n > N_0$).
  2. [Balas, JMAA'88] - qualitative result:
     for large enough "residual mode filter" dimension.
  3. [Harkort & Deutscher, IJC'11] - 1st step to quantitative results:
     conservative estimates on "output filter" and difficult to compute.

- Delayed observer-based control via modal decomposition:
  1. [Katz & Fridman & Selivanov, TAC'21] - PDE observer (separation).

Our goal:
Easily verifiable and efficient conditions for finite-dimensional observer-based controller.

Finite-dim. observer-based control - observer design

$$\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t), \quad n = 1, 2, \ldots$$

Let $\delta > 0$ be a desired decay rate. Let $N_0 \in \mathbb{N}$ satisfy

$$-\lambda_n + q < -\delta, \quad n > N_0.$$

N_0 - controller dimension,
$N \geq N_0$ - observer dimension.

- Finite-dimensional observer: $\hat{z}(x,t) := \sum_{n=1}^{N} \hat{z}_n(t)\phi_n(x)$

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q)\hat{z}_n(t) + b_n u(t) - \ell_n \left[\sum_{n=1}^{N} \hat{z}_n(t)\phi_n(0) - y(t)\right],$$
$$\hat{z}_n(0) = 0, \quad 1 \leq n \leq N.$$

Gains selection

Observer and controller gains are designed independently based on N_0 modes:

- Observer: Let

$$A_0 = \mathrm{diag}\left\{-\lambda_1 + q, \dots, -\lambda_{N_0} + q\right\}, \; L_0 = [l_1, \dots, l_{N_0}]^T,$$
$$C_0 = [c_1, \dots, c_{N_0}], \; c_n = \phi_n(0), \quad n \geq 1.$$

Since $c_n \neq 0$ for $1 \leq n \leq N_0$, (A_0, C_0) is observable with L_0 found from

$$P_{\mathrm{o}}(A_0 - L_0 C_0) + (A_0 - L_0 C_0)^T P_{\mathrm{o}} < -2\delta P_{\mathrm{o}}, \quad P_{\mathrm{o}} > 0.$$

Choose $l_n = 0, \; n > N_0$.

- Controller: Assume $b_n = \langle b, \phi_n \rangle \neq 0$ for $1 \leq n \leq N_0$. Let

$$B_0 := \begin{bmatrix} b_1 & \dots & b_{N_0} \end{bmatrix}^T.$$

Then (A_0, B_0) is controllable. Let $K_0 \in \mathbb{R}^{1 \times N_0}$ satisfy

$$P_{\mathrm{c}}(A_0 + B_0 K_0) + (A_0 + B_0 K_0)^T P_{\mathrm{c}} < -2\delta P_{\mathrm{c}}, \quad P_{\mathrm{c}} > 0$$

Control law and estimation error

We propose a N_0-dimensional controller:

$$u(t) = K_0 \hat{z}^{N_0}(t), \quad \hat{z}^{N_0}(t) = [\hat{z}_1(t), \ldots, \hat{z}_{N_0}(t)]^T$$

based on the N-dimensional observer.

Let $e_n(t) = z_n(t) - \hat{z}_n(t), \; 1 \leq n \leq N$. The error equations can be presented as:

$$\dot{e}_n(t) = (-\lambda_n + q) z_n(t) - l_n \Big(\sum_{n=1}^{N} c_n e_n(t) + \underbrace{\zeta(t)}_{z(0,t) - \sum_{n=1}^{N} c_n z_n(t)} \Big), \; 1 \leq n \leq N.$$

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Denote

$$\begin{aligned}
&e^{N_0}(t) = [e_1(t), \ldots, e_{N_0}(t)]^T, \\
&e^{N-N_0}(t) = [e_{N_0+1}(t), \ldots, e_N(t)]^T, \\
&\hat{z}^{N-N_0}(t) = [\hat{z}_{N_0+1}(t), \ldots, \hat{z}_N(t)]^T, \\
&\mathcal{L} = \text{col}\left\{ L_0, -L_0, 0_{2(N-N_0)\times 1} \right\}, \\
&\tilde{K} = \left[K_0, \quad 0_{1\times(2N-N_0)} \right], \\
&A_1 = \text{diag}\{-\lambda_{N_0+1} + q, \ldots, -\lambda_N + q\}, \\
&C_1 = [c_{N_0+1}, \ldots, c_N], \quad B_1 = [b_{N_0+1}, \ldots, b_N]^T.
\end{aligned}$$

Finite-dim. observer-based control - closed-loop system

Closed-loop system for $t \geq 0$:

$$\dot{X}(t) = FX(t) + \mathcal{L}\zeta(t),$$
$$\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n\tilde{K}X(t), \quad n > N,$$

where

$$X(t) = \text{col}\left\{\hat{z}^{N_0}(t), e^{N_0}(t), \hat{z}^{N-N_0}(t), e^{N-N_0}(t)\right\} \in \mathbb{R}^{2N},$$
$$F = \begin{bmatrix} A_0 + B_0K_0 & L_0C_0 & 0 & L_0C_1 \\ 0 & A_0 - L_0C_0 & 0 & -L_0C_1 \\ B_1K_0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}.$$

Spillover - coupling between finite-dimensional and infinite-dimensional parts

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Spillover - coupling between finite-dimensional and infinite-dimensional parts

We have

$$\begin{aligned} \zeta^2(t) &= \left[z(0,t) - \textstyle\sum_{n=1}^{N}\phi_n(0)z_n(t)\right]^2 \\ &\leq \left\|z_x(\cdot,t) - \textstyle\sum_{n=1}^{N}\phi_n'(\cdot)z_n(t)\right\|^2 = \textstyle\sum_{n=N+1}^{\infty}\lambda_n z_n^2(t) \end{aligned}$$

Finite-dim. observer-based control - Stability analysis

For H^1-stability we use

$$V(t) = X^T(t)PX(t) + \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t), \quad 0 < P \in \mathbb{R}^{2N\times 2N}.$$

Differentiating along the closed-loop system:

$$\dot{V} + 2\delta V = X^T(t)\left[PF + F^TP + 2\delta P\right]X(t) + 2X^T(t)P\mathcal{L}\zeta(t)$$

$$+2\sum_{n=N+1}^{\infty} \lambda_n(-\lambda_n + q + \delta)z_n^2(t) + \sum_{n=N+1}^{\infty} 2z_n(t)\lambda_n b_n \tilde{K}X(t).$$

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We apply Young's inequality to the cross terms:

$$\sum_{n=N+1}^{\infty} 2\lambda_n z_n(t) b_n \tilde{K}X(t) \leq \frac{1}{\alpha}\sum_{n=N+1}^{\infty} \lambda_n z_n^2(t) + \alpha \left\|b'\right\|_{L^2}^2 \left\|\tilde{K}X(t)\right\|^2.$$

Then

$$2\sum_{n=N+1}^{\infty} \lambda_n\left(-\lambda_n + q + \delta + \frac{1}{2\alpha}\right)z_n^2(t) \leq -2\left(\lambda_{N+1} - q - \delta - \frac{1}{2\alpha}\right)\zeta^2(t)$$

Finite-dim. observer-based control - Stability analysis

Let $\eta(t) = \text{col}\left\{X(t), \zeta(t)\right\}$. The stability analysis leads to

$$\dot{V} + 2\delta V \leq \eta^T(t)\Phi\eta(t) \leq 0$$

provided

$$\Phi = \begin{bmatrix} PF + F^T P + 2\delta P + \alpha \left\|b'\right\|^2 \tilde{K}^T \tilde{K} & P\mathcal{L} \\ * & -2\left(\lambda_{N+1} - q - \delta - \frac{1}{2\alpha}\right) \end{bmatrix} < 0.$$

Can be converted to LMI by Schur complement.

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Observations:

- The LMI dimension grows with N
- $\|P\|$ can grow - may lead to infeasibility for all $N \in \mathbb{N}$

Our contribution:

- Derivation of constructive LMI condition.
- Proof of feasibility for large N
  (based on asymptotic perturbation analysis to bound $\|P\|$).

Finite-dim. observer-based control - Stability analysis

- Summarizing:

Given $\delta > 0$, if there exist $0 < P \in \mathbb{R}^{2N \times 2N}$ and $\alpha > 0$ that satisfy the LMI, then

$$\|z(\cdot,t)\|_{H^1}^2 + \|z(\cdot,t) - \hat{z}(\cdot,t)\|_{H^1}^2 \leq Me^{-2\delta t} \|z_0\|_{H^1}^2 ,$$

with some constant $M > 0$. Moreover, the LMI is always feasible for large enough N.

Finite-dim. observer-based control - Stability analysis

- Summarizing:

  Given $\delta > 0$, if there exist $0 < P \in \mathbb{R}^{2N \times 2N}$ and $\alpha > 0$ that satisfy the LMI, then
  $$\|z(\cdot,t)\|^2_{H^1} + \|z(\cdot,t) - \hat{z}(\cdot,t)\|^2_{H^1} \leq Me^{-2\delta t} \|z_0\|^2_{H^1} ,$$
  with some constant $M > 0$. Moreover, the LMI is always feasible for large enough N.

Other cases treated in [Katz & Fridman, Aut '20] :

→ Non-local measurement and actuation - L^2 and H^1 stability

→ Dirichlet actuation and non-local measurement - $H^{-\frac{1}{2}}$ stability $(V = \sum \lambda_n^{-1} z_n^2)$
In this case,
$$|b_n| \approx \sqrt{\lambda_n}$$
which is difficult to compensate in the Lyapunov analysis even for the L^2-norm.

Point measurement & actuation - dynamic extension

[Katz & Fridman, CDC '20; TAC '22]

Kuramoto-Sivashinsky equation (KSE)

$$\begin{aligned} z_t(x,t) \quad &= -z_{xxxx}(x,t) - \nu z_{xx}(x,t), \quad t \geq 0, \\ &z(0,t) = u(t), \quad z(1,t) = 0, \\ &z_{xx}(0,t) = 0, \quad z_{xx}(1,t) = 0. \end{aligned}$$

Measurement : $y(t) = z(x_*, t), \quad x_* \in (0,1)$

- Mixed Dirichlet boundary conditions.
- Point measurement and boundary actuation - unbounded operators.

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Dynamic extension [Curtain & Zwart, 95], [Prieur & Trélat, Aut '18], [Katz & Fridman, Aut '21]:

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) := 1 - x$$

Results in better behaved $\{b_n\}_{n=1}^{\infty} \Rightarrow$ convergence in stronger norms.

Point measurement & actuation - dynamic extension

Existing results on KSE:

- Distributed state-feedback/observer-based control via modal decomposition [Christofides & Armaou. SCL '00]
- Boundary control, small anti-diffusion [Liu & Krstić. Nonlin Analysis. '01]
- State-feedback stabilization of KSE under boundary/non-local actuation [Cerpa. Commun. Pure Appl. Anal, '10], [Cerpa, Guzman & Mercado. ESAIM, '17], [Guzman, Marx & Cerpa. CPDE '19]
  - $\rightarrow$ Different boundary conditions $\Rightarrow$ no explicit estimates on eigenvalues and eigenfunctions
  - $\rightarrow$ Theoretically possible but computationally expensive

Point measurement & actuation - dynamic extension

Equivalent ODE-PDE system:

$$\dot{u}(t) = v(t), \quad w_t(x,t) = -w_{xxxx}(x,t) - \nu w_{xx}(x,t) - r(x)v(t)$$

with

$$\begin{aligned} &u(0) = 0, \\ &w(0,t) = 0, \quad w(1,t) = 0, \\ &w_{xx}(0,t) = 0, \quad w_{xx}(1,t) = 0. \end{aligned}$$

- New measurement: $y(t) = w(x_*,t) + r(x_*)u(t)$.
- $u(t)$ - additional state, $v(t)$ - control input
- Given $v(t)$, $u(t)$ is computed by

$$\dot{u}(t) = v(t), \quad u(0) = 0$$

Modal decomposition using Sturm-Liouville operator for KSE:

$$\lambda_n = \pi^2 n^2, \ \phi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n} x), \quad n \geq 1$$

Point measurement & actuation - modal decomposition

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x)$$

$$\Downarrow$$

$$\dot{w}_n(t) = (-\lambda_n^2 + \nu\lambda_n)w_n(t) + b_n v(t),\ \ w_n(0) = \langle z_0, \phi_n\rangle\,,$$

$b_n = -\sqrt{\frac{2}{\lambda_n}}$ $\quad$ $\ell^2(\mathbb{N})$ sequence, nonzero elements.

Point measurement & actuation - modal decomposition

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x)$$

$$\Downarrow$$

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Lyapunov H^1-stability analysis leads to LMIs:

$$\begin{bmatrix} PF + F^T P + 2\delta P + \frac{2\alpha}{\pi^2 N}\tilde{K}^T\tilde{K} & P\mathcal{L} \\ * & -\beta \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\lambda_{N+1} + \nu + \frac{2\delta+\beta}{2\lambda_{N+1}} & \frac{1}{\sqrt{2}} \\ * & -\alpha \end{bmatrix} < 0.$$

where $P > 0$ is a matrix and $\alpha, \beta > 0$ are scalars.

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Feasibility of the LMIs leads to

$$\|w(\cdot,t)\|_{H^1} + |u(t)| + \|w(\cdot,t) - \hat{w}(\cdot,t)\|_{H^1} \leq Me^{-\delta t}\,\|w(\cdot,0)\|_{H^1}\,.$$

with some constant $M > 0$.

L^2-gain and ISS analysis

In [Katz & Fridman,TAC '22] we consider

$$\begin{aligned} &z_t(x,t) = -z_{xxxx}(x,t) - \nu z_{xx}(x,t) + d(x,t), \\ &z(0,t) = u(t), \quad z(1,t) = 0, \quad z_{xx}(0,t) = z_{xx}(1,t) = 0 \end{aligned}$$

with in-domain point measurement

$$y(t) = z(x_*,t) + \sigma(t), \; x_* \in (0,1).$$

The disturbances satisfy

$$\begin{aligned} &d \in L^2((0,\infty); L^2(0,1)) \cap H^1_{\mathrm{loc}}((0,\infty); L^2(0,1)), \\ &\sigma \in L^2(0,\infty) \cap H^1_{\mathrm{loc}}(0,\infty). \end{aligned}$$

Dynamic extension:

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) := 1 - x$$

L^2-gain and ISS analysis

Let $\gamma > 0$ and $\rho_w, \rho_u \geq 0$ be scalars. We introduce the performance index

$$J = \int_0^\infty \left[\rho_w^2 \left\| w(\cdot,t) \right\|_{L^2}^2 + \rho_u^2 u^2(t) - \gamma^2 \left(\| d(\cdot,t) \|_{L^2}^2 + \sigma^2(t) \right) \right] dt.$$

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We find conditions that guarantee along the closed-loop

$$\begin{aligned}
&\dot{V} + 2\delta V + W \leq 0,\\
&W = \rho_w^2 \left\|w(\cdot,t)\right\|_{L^2}^2 + \rho_u^2 u^2(t) - \gamma^2 \left(\|d(\cdot,t)\|_{L^2}^2 + \sigma^2(t)\right),\\
&V(t) = |X_N(t)|_P^2 + \sum_{n=N+1}^\infty \lambda_n w_n^2(t)
\end{aligned}$$

$$\Downarrow$$

- $\delta = 0 \Rightarrow J \leq 0$
- $\delta > 0$ and $\rho_w = \rho_u = 0 \Rightarrow$ ISS, i.e. for some $\overline{M} > \underline{M} > 0$:

$$\begin{aligned}
\underline{M} \quad \left[|u(t)|^2 + \|w(\cdot,t)\|_{H^1}^2\right] \leq & \overline{M} e^{-2\delta T} \left\|w(\cdot,0)\right\|_{H^1}^2 \\
& + \frac{\gamma^2}{2\delta} \sup_{0\leq t\leq T} \left[\|d(\cdot,t)\|_{L^2}^2 + \sigma^2(t)\right] \quad \forall T > 0,
\end{aligned}$$

L^2-gain and ISS analysis

Our L^2-gain analysis results in the following LMI:

$$\Psi_N^{(1)} = \left[\begin{array}{cc|cc} \Phi_N^{(1)} + \Xi & P\mathcal{L} & P & P\mathcal{L} \\ * & -2\left(\theta_{N+1}^{(1)} - \frac{\lambda_{N+1}}{2\alpha} - \frac{\lambda_{N+1}}{2\gamma^2}\right) & 0 & 0 \\ \hline & * & & -\gamma^2 I \end{array}\right] < 0,$$

$$\Phi_N^{(1)} = PF + F^T P + 2\delta P + \frac{2\alpha}{\pi^2 N}\tilde{K}_0^T \tilde{K}_0,$$

$$\Xi = \Xi_1^T \Xi_1, \ \Xi_1 = \begin{bmatrix} \rho_u & 0 & 0 & 0 & 0 \\ 0 & \rho_w I_{N0} & \rho_w I_{N0} & 0 & 0 \\ 0 & 0 & 0 & \rho_w I_{N-N_0} & \rho_w I_{N-N_0} \end{bmatrix}.$$

Novelty: proof of the LMI feasibility for large enough γ and N

- Ξ: positive term, which is not multiplied by a decision variable and does not decay with N (compare with $\frac{2\alpha}{\pi^2 N}\tilde{K}^T\tilde{K}$)
- For ISS with $d(x,t) \equiv 0$, the LMI feasibility for N implies its feasibility for $N+1$. Thus, increasing N does not deteriorate the performance.

Reduced-order LMIs

[Katz et al, ECC '21 & Aut, under review]

Consider heat equation with Neumann actuation

$$\begin{aligned} z_t(x,t) &= z_{xx}(x,t) + qz(x,t), \\ z_x(0,t) &= 0, \quad z_x(1,t) = u(t). \end{aligned}$$

Non-local measurement

$$y(t) = \langle c, z(\cdot,t) \rangle, \quad c \in L^2(0,1).$$

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$$y(t) = \langle c, z(\cdot,t)\rangle, \quad c \in L^2(0,1).$$

- No dynamic extension for L^2-stability:

$\rightarrow$ $\lambda_n = \pi^2 n^2,\ n \geq 0\ ;\ \phi_0(x) = 1,\ \phi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x), \quad n \geq 1$

$$\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t), t \geq 0,$$
$$b_0 = 1, \quad b_n = (-1)^n\sqrt{2}, \quad - \ell^\infty(\mathbb{N})$$

$\rightarrow$ The estimation error tail $\zeta(t)$ satisfies

$$\zeta^2(t) \leq \underbrace{\|c\|_N^2}_{\sum_{n=N+1} c_n^2 \overset{N\to\infty}{\to} 0} \sum_{n=N+1}^{\infty} z_n^2(t),$$

Reduced-order closed-loop system

The reduced-order closed-loop system is given by

$$\dot{X}_0(t) = F_0 X_0(t) + \mathcal{L}_0 C_1 e^{N-N_0}(t) + \mathcal{L}_0 \zeta(t),$$
$$\dot{z}_n(t) = (-\lambda_n + q) z_n(t) + b_n \mathcal{K}_0 X_0(t), \;\; n > N.$$

where

$$F_0 = \begin{bmatrix} A_0 + B_0 K_0 & L_0 C_0 \\ 0 & A_0 - L_0 C_0 \end{bmatrix},$$
$$X_0(t) = \text{col}\left\{ \hat{z}^{N_0}(t), e^{N_0}(t) \right\}.$$

What about $\hat{z}^{N-N_0}(t)$ and $e^{N-N_0}(t)$?

$\dot{\hat{z}}^{N-N_0}(t) = A_1 \hat{z}^{N-N_0}(t) + B_1 \mathcal{K}_0 X_0(t) \Rightarrow$ exp. decaying provided $X_0(t)$ is

$\dot{e}^{N-N_0}(t) = A_1 e^{N-N_0}(t) \Rightarrow$ exp. decaying

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$$\dot{e}^{N-N_0}(t) = A_1 e^{N-N_0}(t) \Rightarrow \text{exp. decaying}$$

- Advantages of the reduced-order closed-loop:
  - → Takes into account the fast-slow structure of the dynamics
  - → Reduced-order LMIs, which are more computationally efficient
  - → Trivializes proofs of LMIs feasibility for large N, and of feasibility for $N \Rightarrow N+1$

Stability analysis

For L^2-stability we use

$$V(t) = V_0(t) + p_e \left| e^{N-N_0}(t) \right|^2, \; V_0(t) = |X_0(t)|^2_{P_0} + \sum_{n=N+1}^{\infty} z_n^2(t)$$

where $0 < P \in \mathbb{R}^{(2N_0+1)\times(2N_0+1)}$, $p_e \to \infty$ leading to the reduced-order LMI:

$$\begin{bmatrix} \Phi_0 & P_0\mathcal{L}_0 & 0 \\ * & -2\left(\lambda_{N+1} - q - \delta\right) \|c\|_N^{-2} & 1 \\ * & * & -\frac{\alpha\|c\|_N^2}{\lambda_{N+1}} \end{bmatrix} < 0,$$

$$\Phi_0 = P_0F_0 + F_0^TP_0 + 2\delta P_0 + \frac{2\alpha}{\pi^2 N}\mathcal{K}_0^T\mathcal{K}_0.$$

$\rightarrow$ The LMI dimension does not grow with N

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$\rightarrow$ The LMI dimension does not grow with N

- In the numerical example we easily verify LMIs for $N = 30$, whereas feasibility of the full-order LMIs could be verified for $N \leq 9$.
- Since we don't use dynamic extension, we can treat general time-varying delays & sampled-data control via a ZOH
- To enlarge delays, in [Katz & Fridman, Aut, under review] we compensate constant part of an input delay via classical predictor.

Plan

1. Introduction: delays, spatial & modal decomposition
2. Constructive finite-dimensional observer-based control
3. Delayed implementation
4. Sampled-data implementation
5. Semilinear PDEs

Delayed implementation - problem formulation

[Katz & Fridman, Aut '21]

$$
\begin{aligned}
&z_t(x,t) = z_{xx}(x,t) + qz(x,t) + b(x)u(t-\tau_u(t)),\\
&z_x(0,t) = 0,\ \ z(1,t) = 0,
\end{aligned}
$$

$$
y(t) = z(0, t-\tau_y(t))
$$

Consider $b \in H^1(0,1),\ b(1) = 0.$

- $\tau_y(t)$ - known measurement delay, $\tau_y(t) \leq \tau_M$
- $\tau_u(t)$ - unknown input delay, $\tau_u(t) \leq \tau_M$
- C^1 delays or sawtooth delays (correspond to sampled-data or networked control)

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- C^1 delays or sawtooth delays (correspond to sampled-data or networked control)

$$\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t-\tau_u(t)),$$
$$z_n(0) = \langle z_0, \phi_n \rangle =: z_{0,n}, \quad b_n = \langle b, \phi_n \rangle .$$

Let $N_0 \in \mathbb{N}$ satisfy

$$-\lambda_n + q < -\delta_\tau, \quad n > N_0.$$

N_0 - the controller dimension. $N \ge N_0$ - the observer dimension.

Delayed implementation - observer design

- Finite-dimensional observer: $\hat{z}(x,t) := \sum_{n=1}^{N} \hat{z}_n(t)\phi_n(x)$.

$$\begin{aligned}&\dot{\hat{z}}_n(t) = (-\lambda_n + q)\hat{z}_n(t) + b_n u(t) - \ell_n \left[\sum_{n=1}^{N} c_n \hat{z}_n(t-\tau_y(t)) - y(t)\right],\\ &\hat{z}_n(t) = 0, \quad t \le 0, \quad c_n = \phi_n(0) = \sqrt{2}, \quad 1 \le n \le N.\end{aligned}$$

  $\{\ell_n\}_{n=1}^{N}$ - scalar observer gains.

- Controller: $u(t) = K_0 \hat{z}^{N_0}(t)$.

- Closed-loop system for $t \ge 0$:

$$\dot{X}(t) = FX(t) + F_1 X(t-\tau_y(t)) + F_2 \tilde{K} X(t-\tau_u(t)) + \mathcal{L}\zeta(t-\tau_y(t)),$$

$$\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n \tilde{K} X(t-\tau_u(t)), \ n > N.$$

$$\zeta^2(t-\tau_y(t)) \le \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t-\tau_y(t))$$

Delayed implementation - closed-loop system

We use Lyapunov functional for H^1-stability

$$V(t) = V_{\mathsf{nom}}(t) + \sum_{i=1}^{2} V_{S_i}(t) + \sum_{i=1}^{2} V_{R_i}(t),$$
$$V_{\mathsf{nom}}(t) = X^T(t)PX(t) + \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t),$$

- $V_{S_i}(t)$ and $V_{R_i}(t)$ compensate delays in $X(t)$

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Theorem (Halanay's inequality)

Let $0 < \delta_1 < \delta_0$ *and* $V : [-\tau, \infty) \longrightarrow [0, \infty)$ *be an absolutely continuous s.t.*

$$\dot{V}(t) + 2\delta_0 V(t) - 2\delta_1 \sup_{-\tau \le \theta \le 0} V(t+\theta) \le 0, \quad t \ge 0.$$

Then $V(t) \le e^{-2\delta_\tau t} \sup_{-\tau \le \theta \le 0} V(\theta),\ t \ge 0$ *where* $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau \tau}$.

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$$-2\delta_1 \sup_{-\tau_M \le \theta \le 0} V(t+\theta) \le -2\delta_1 \, |X(t - \tau_y(t))|_P^2 - 2\delta_1 \zeta^2(t - \tau_y(t))$$

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$$-2\delta_1 \sup_{-\tau_M \le \theta \le 0} V(t+\theta) \le -2\delta_1 \left|X(t-\tau_y(t))\right|_P^2 - 2\delta_1 \zeta^2(t-\tau_y(t))$$

- We prove: the resulting LMIs are feasible for large N and small τ_M.

Predictor/Subpredictors

Q: What about large delay compensation?

In [Katz & Fridman, L-CSS '21] we consider

$$z_t(x,t) = z_{xx}(x,t) + qz(x,t),\ x \in [0,1],\ t \geq 0,$$
$$z_x(0,t) = 0, \quad z_x(1,t) = u(t-r)$$

with known delay r and

$$y(t) = \langle c, z(\cdot,t) \rangle,\ t \geq 0, \quad c \in L^2(0,1)$$

Challenge:
Observer-based L^2-stabilization for arbitrarily large delay r via efficient reduced-order LMIs.

Predictor/Subpredictors via reduced-order LMIs

To compensate r we employ a chain of M sub-predictors

$$\hat{z}_1^{N_0}(t-r) \mapsto \cdots \mapsto \hat{z}_i^{N_0}\left(t - \frac{M-i+1}{M} r\right) \mapsto \cdots \mapsto \hat{z}_M^{N_0}(t - \tfrac{r}{M}) \mapsto z^{N_0}(t)$$

Here $\hat{z}_M^{N_0}(t)$ predicts the value of $z^{N_0}(t + \frac{r}{M})$.

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Here $\hat{z}_M^{N_0}(t)$ predicts the value of $z^{N_0}(t + \frac{r}{M})$.

Intuition: $\hat{z}_1^{N_0}(t) \approx z^{N_0}(t+r) \Rightarrow u(t-r) \approx -K_0 z^{N_0}(t)$.

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Novelty: Closed-loop system for $t \geq 0$ is given by

$$\begin{aligned}
\dot{z}^{N_0}(t) &= (A_0 - B_0K_0)z^{N_0}(t) + B_0\mathcal{K}_e X_e(t) \\
\dot{X}_e(t) &= F_e X_e(t) + G_e\left[X_e\left(t - \tfrac{r}{M}\right) - X_e(t)\right] + \mathcal{L}_e\zeta\left(t - \tfrac{r}{M}\right) \\
&\quad + \mathcal{L}_e C_1 e^{-A_1\frac{r}{M}} e^{N-N_0}(t), \\
\dot{z}_n(t) &= (-\lambda_n + q)z_n(t) - b_n K_0 z^{N_0}(t), \\
&\quad + b_n\mathcal{K}_e X_e(t), \quad n > N.
\end{aligned}$$

Predictor/Subpredictors via reduced-order LMIs

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\dot{X}_e(t) &= F_e X_e(t) + G_e \left[X_e\left(t - \tfrac{r}{M}\right) - X_e(t) \right] + \mathcal{L}_e \zeta\left(t - \tfrac{r}{M}\right) \\
&\quad + \mathcal{L}_e C_1 e^{-A_1 \frac{r}{M}} e^{N-N_0}(t), \\
\dot{z}_n(t) &= (-\lambda_n + q) z_n(t) - b_n K_0 z^{N_0}(t), \\
&\quad + b_n \mathcal{K}_e X_e(t), \quad n > N.
\end{aligned}$$

- Closed-loop system includes the state $z^{N_0}(t)$ (not $\hat{z}^{N_0}(t)$), subpredictor estimation errors $X_e(t)$ and tail $z_n(t), n > N$
- The formulation eliminates r from ODEs of $z^{N_0}(t)$ and $z_n(t),\ n > N$ and decreases it to $\frac{r}{M}$ in $X_e(t)$.
- Reduced-order closed-loop system

Predictor/Subpredictors via reduced-order LMIs

We carry out L^2-stability analysis of the reduced-order closed-loop system, leading to reduced-order LMIs.

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- By induction, we construct a Lyapunov function for the subpredictor errors, taking into account the cascaded structure of the ODEs.
- Choose the remaining decision variables, which depend on N and take $N \to \infty$ to show feasibility of the LMIs

Predictor/Subpredictors via reduced-order LMIs

We also consider compensation of r using a classical predictor:

$$\bar{z}(t) = e^{A_0 r}\hat{z}^{N_0}(t) + \int_{t-r}^{t} e^{A_0(t-s)}B_0 u(s)ds, \; u(t) = -K_0\bar{z}(t)$$

The resulting reduced-order closed-loop system consists of ODEs for $\bar{z}(t)$, $e^{N_0}(t)$ and $z_n(t), \; n > N$.

Lyapunov L^2-stability analysis leads to reduced-order LMI.

For the case of a classical predictor, we prove LMIs feasibility for arbitrary constant delays provided observer dimension is large.

Plan

Sampled-data implementation via dynamic extension

In [Katz & Fridman, Aut '21], we consider:

$$\begin{array}{l} z_t(x,t) = z_{xx}(x,t) + az(x,t), \ \ t \geq 0, \\ z_x(0,t) = 0, \quad z(1,t) = u(t) \end{array}$$

in the presence of two independent communication networks.

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Sampled-data in measurements:

- Sampling instances $0 = s_0 < s_1 < \cdots < s_k < \ldots, \lim_{k\to\infty} s_k = \infty$

$$s_{k+1} - s_k \leq \tau_{M,y},\ \forall k \in \mathbb{Z}_+, \quad \tau_{M,y} > 0.$$

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- Quantizer $q : \mathbb{R} \to \mathbb{R}$

$$|q[r] - r| \leq \Delta, \quad \text{for all } r \in \mathbb{R}$$

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$$y(t) = q\left[z(x_*, s_k)\right],\ x_* \in [0,1),\ t \in [s_k, s_{k+1}).$$

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Remark: We consider H^1-ISS analysis of the closed-loop systems $\Rightarrow$ possible to also consider saturation.

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Sampled-data in actuation:

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- Controller holding times $0 = t_0 < t_1 < \cdots < t_j < \ldots$, $\lim_{j\to\infty} t_j = \infty$

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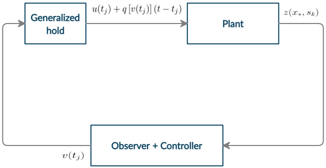
Generalized hold - given $v(t_j)$, the control signal is computed as:

$$u(t) = u(t_j) + q[v(t_j)](t - t_j),\ t \in [t_j, t_{j+1}),\ j = 0, 1, \ldots$$



Sampled-data implementation via dynamic extension

Dynamic extension:

$$w(x,t) = z(x,t) - u(t)$$

leads to the equivalent ODE-PDE system

$$\begin{aligned}\dot{u}(t) &= q\left[v(t_j)\right], \quad t \in [t_j, t_{j+1}),\\ w_t(x,t) &= w_{xx}(x,t) + aw(x,t) + au(t) - q\left[v(t_j)\right],\end{aligned}$$

with homogeneous boundary conditions and

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$(N_0 + 1)$-dimensional observer-based controller

$$\begin{aligned}
\dot{u}(t) &= q\left[v(t_j)\right], \quad t \in [t_j, t_{j+1}), \\
v(t_j) &= -K_0 \hat{w}^{N_0}(t_j), \\
\hat{w}^{N_0}(t) &= [u(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)]^T
\end{aligned}$$

Sampled-data implementation via dynamic extension

Reduced-order closed-loop system for $t \geq 0$:

$$\begin{aligned}
\dot{X}_0(t) \quad &= F_0 X_0(t) + \mathcal{L}\mathcal{C}\Upsilon_y(t) - \mathcal{B}\tilde{K}_0\Upsilon_u(t) + \mathcal{B}\sigma_u(t) \\
&+\mathcal{L}C_1 e^{-A_1\tau_y} e^{N-N_0}(t) + \mathcal{L}\zeta(t-\tau_y) + \mathcal{L}\sigma_y(t), \\
\dot{w}_n(t) \quad &= (-\lambda_n + a)w_n(t) + b_n\left[\tilde{K}_a X_0(t) + \tilde{K}_0\Upsilon_u(t)\right] \\
&-b_n\sigma_u(t), \;\; n > N, \quad t \geq 0
\end{aligned}$$

Here

$$\tau_y(t) = t - s_k, \quad t \in [s_k, s_{k+1}), \quad \tau_y(t) \leq \tau_{M,y}$$

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The quantization errors

$$\begin{aligned}
\sigma_y(t) &= q\left[w(x_*, t-\tau_y) + u(t-\tau_y)\right] \\
&\quad - w(x_*, t-\tau_y) - u(t-\tau_y), \\
\sigma_u(t) &= q\left[-K_0\hat{w}^{N_0}(t_j)\right] + K_0\hat{w}^{N_0}(t_j), \ t \in [t_j, t_{j+1}).
\end{aligned}$$

are treated as disturbances

$$\max\left(\|\sigma_u\|_\infty, \|\sigma_y\|_\infty\right) \leq \Delta.$$

Sampled-data implementation via dynamic extension

For H^1-ISS analysis, we use a Wirtinger-based Lyapunov functional - efficient for sampled-data control

<u>Challenge:</u>
$V(t)$ may have jump discontinuities at $s_k,\ k \in \mathbb{Z}_+$ and inside the intervals $[s_k, s_{k+1})$, where we want to apply Halanay's inequality.

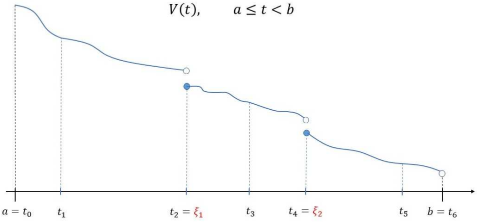

Figure 2: Possible behavior of $V(t)$

Sampled-data implementation via dynamic extension

We prove a novel form of Halanay's inequality for ISS

Theorem

Let $V : [a,b) \to [0,\infty)$ *be a bounded function, where* $b-a \le h$ *for* $h>0$. *Assume* $V(t)$ *is continuous on* $[t_i, t_{i+1}),\ i=0,\dots,N-1$, *where*

$$a =: t_0 < t_1 < \cdots < t_{N-1} < t_N := b,$$

and

$$\lim_{t \nearrow t_i} V(t) \ge V(t_i), \quad i = 1,2,\dots,N-1.$$

Assume further that for some $d \ge 0$ *and* $\delta_0 > \delta_1 > 0$

$$D^+V(t) \le -2\delta_0 V(t) + 2\delta_1 \sup_{a\le\theta\le t} V(\theta) + d,\ \ t \in [a,b)$$

where $D^+V(t)$ *is the right upper Dini derivative, defined by*

$$D^+V(t) = \limsup_{s\to 0^+} \frac{V(t+s)-V(t)}{s}.$$

Then

$$V(t) \le e^{-2\delta_\tau (t-a)} V(a) + d\int_a^t e^{-2\delta(t-s)} ds,\ \ t\in[a,b)$$

where $\delta = \delta_0 - \delta_1$ *and* $\delta_\tau > 0$ *solves* $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau h}$.

Sampled-data implementation via dynamic extension

H^1-ISS analysis leads to Reduced-order LMIs for ISS

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Feasibility of the LMIs guarantees

$$\|w(\cdot,t)\|_{H^1}^2 + \|\hat{w}(\cdot,t)\|_{H^1}^2 + u^2(t) \\ \leq M_0 e^{-2\delta_\tau t} \|w(\cdot,0)\|_{H^1}^2 + r^2 \Delta^2, \quad t \geq 0$$

- r - explicitly estimated in the analysis

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The LMIs are always feasible for large enough N and small enough $\tau_{M,y}, \tau_{M,u}$, their feasibility for N implies feasibility for $N+1$.

Plan

Semilinear PDEs

Q: Can our approach be generalized to nonlinear equations?

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In [Katz & Fridman, L-CSS '22] we consider regional stabilization of

$$\begin{aligned}
&z_t(x,t) = -z_{xxxx}(x,t) - \nu z_{xx}(x,t) - \frac{1}{2}\left(z^2(x,t)\right)_x, \\
&z(0,t) = 0, \quad z(1,t) = u(t), \quad z_{xx}(0,t) = 0, \quad z_{xx}(1,t) = 0
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dynamic extension: Let $\kappa > 0$

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) = x$$

leads to

$$\begin{aligned} &\dot{u}(t) = -\kappa u(t) + v(t), \ u(0) = 0, \\ &w_t(x,t) = -w_{xxxx}(x,t) - \nu w_{xx}(x,t) + \kappa r(x)u(t) \\ &\qquad\qquad -r(x)v(t) - [w(x,t) + xu(t)]\,[w_x(x,t) + u(t)], \\ &w(0,t) = w(1,t) = w_{xx}(0,t) = w_{xx}(1,t) = 0. \end{aligned}$$

Semilinear PDEs

Modal decomposition: $w(x,t) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x)$

$$\begin{aligned}
\dot{w}_n(t) &= \left(-\lambda_n^2 + \nu\lambda_n\right) w_n(t) + \kappa b_n u(t) - b_n v(t) \\
&\quad - w_n^{(1)}(t) - w_n^{(2)}(t), \ \ t \geq 0, \\
w_n^{(1)}(t) &= \langle [w(\cdot,t) + \cdot u(t)]\, w_x(\cdot,t), \phi_n \rangle, \\
w_n^{(2)}(t) &= \langle w(\cdot,t) + \cdot u(t), \phi_n \rangle\, u(t)
\end{aligned}$$

Controller:

$$v(t) = -Kw^N(t), \ \ w^N(t) = \mathrm{col}\left\{u(t), w_n(t)\right\}_{n=1}^{N}.$$

Closed-loop system for $t \geq 0$:

$$\begin{aligned}
\dot{w}^N(t) &= (A - BK)w^N(t) - w^{N,(1)}(t) - w^{N,(2)}(t), \\
\dot{w}_n(t) &= \left(-\lambda_n^2 + \nu\lambda_n\right) w_n(t) + \kappa b_n u(t) - b_n v(t) \\
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\end{aligned}$$

Semilinear PDEs

For H^1-stability analysis of the closed-loop system, we consider

$$V(t) = \left|w^N(t)\right|_P^2 + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t),$$

where $0 < P \in \mathbb{R}^{(N+1)\times(N+1)}$.

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To compensate semilinearity, let $0 < \sigma \in \mathbb{R}$ and assume

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We use the Young/Sobolev inequalities and Parseval's equality in the cross terms

$$-2\sum_{n=N+1}^{\infty} \lambda_n w_n(t) w_n^{(1)}(t) \le \alpha_2 \sum_{n=1}^{\infty} \lambda_n^2 w_n^2(t) \\ -\frac{1}{\alpha_2}\left|w^{N,(1)}(t)\right|^2 + \frac{1}{\alpha_2}\sum_{n=1}^{\infty}\left[w_n^{(1)}(t)\right]^2.$$

and

$$\frac{1}{\alpha_2}\sum_{n=1}^{\infty}\left[w_n^{(1)}(t)\right]^2 \overset{\text{Pars.}}{=} \frac{1}{\alpha_2}\int_0^1 [w(x,t)+xu(t)]^2 w_x^2(x,t)dx \\ \overset{\text{Sob.}}{\le} \frac{2\sigma^2}{\alpha_2}\|w_x(\cdot,t)\|^2 = \frac{2\sigma^2}{\alpha_2}\left|w^N(t)\right|_\Lambda^2 + \frac{2\sigma^2}{\alpha_2}\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t)$$

Semilinear PDEs

Our H^1-stability analysis leads to LMIs which

- Involve σ as a tuning parameter
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$$\|w(\cdot,t)\|^2_{H^1} + u^2(t) \leq Me^{-2\delta t}\,\|w(\cdot,0)\|^2_{H^1}\,,\ t \geq 0,$$

The assumption involving $\sigma > 0$ requires an estimate on the domain of attraction, in terms of the original state $z(x,t)$.

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We derive a lower bound $\rho > 0$ such that

- It can be computed explicitly in terms of σ and the LMI decision variables
- If $\|z_x(\cdot,0)\|^2 < \rho^2$ then solution of the closed-loop system exists for all time and is H^1-exp. stable.

Conclusions

A dream about efficient finite-dimensional observer-based control comes true:

a LMI-based method is introduced for parabolic PDEs via modal decomposition.

- → Observer dimension, ISS & L^2-gain, delay bounds are found from LMIs.
- → LMIs are proved to be asymptotically feasible and they are only slightly conservative in examples.
- → LMIs may be verified by users without any background in PDEs!
- → Large input delays are compensated by predictors.
- → For point measurement and actuation via dynamic extension, sampled-data implementation employs generalized hold
- → Our approach can be extended to semilinear parabolic PDEs

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- December 10, 2021
  "TDS seminar" (organizers W. Michiels & G. Orosz)
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